Controlled quantum state transfer in spin chains

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Summary

1. Control
2. State transfer in quantum chains
3. Control in finite Hilbert spaces: general results
4. What is actually done

For a given quantum system whose Hamiltonian depends on some independent control functions $u(t)$, the time evolution equation is given by:

$$|\dot{\psi}\rangle = -iH(u(t))|\psi\rangle \qquad |\psi(t=0)\rangle = |\psi_0\rangle$$

For two given states $|\psi_0\rangle$ and $|\psi_T\rangle$ the problem is:

Is there a time $T < \infty$ and a control function $u(t)$ such that if $|\psi(t=0)\rangle = |\psi_0\rangle$ then $|\psi(t=T)\rangle = |\psi_T\rangle$?

◦ Some examples: or the Hamiltonian

$$H(t) = \gamma\, \mathbf{S}.\mathbf{B}(t) = \gamma\, (S_x B_x(t) + S_y B_y(t) + S_z B_z(t))$$

The control functions are $\mathbf{B}(t)$.

◦ A more general example:

$$H(t) = \sum_{n=1}^{N} E_n |n\rangle\langle n| \; + \; \sum_{m=1}^{M} f_m(t)\, \hat{H}_m$$

Where the control functios are $f_m(t)$.

Transfer in a quantum chain [1]

Alice wants to sent a state $|\psi_{in}\rangle$ to Bob. Their transfer channel is given by:

$$H(t) = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} B_i(t)\sigma_z^i$$

The initial state $|\mathbf{0}\rangle = |000...0\rangle$ where $|0\rangle$ corresponds to the spin down state and we take $E_0 = 0$.
The one excitation states $|\boldsymbol{j}\rangle = |00..10..0\rangle$ (where ***j*=1...*N***) where the spin in the j site is now up.
For simplicity, and because it resembles a “realistic” setup, Alice and Bob have access to the sites s and r.

[1] Sougato Bose *Quantum Communication Through Spin Dynamics: An Introductory Overview*

State to be transferred

$$|\psi_{in}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle.$$

State of the whole chain

$$|\psi(0)\rangle = \cos(\theta/2)|\mathbf{0}\rangle + e^{i\phi}\sin(\theta/2)|\mathbf{s}\rangle$$

Bob hopes that after some time the state of the chain will be

$$\cos(\theta/2)|\mathbf{0}\rangle + e^{i\phi}\sin(\theta/2)|\mathbf{r}\rangle.$$

Note that

$$\left[H_G, \sum_{i=1}^{N} \sigma_z^i\right] = 0,$$

so the time evolution preserves the total magnetization

At time *t* the state is given by

$$|\psi(t)\rangle = \cos(\tfrac{\theta}{2})|\mathbf{0}\rangle + e^{i\phi}\sin(\tfrac{\theta}{2})\sum_{j=1}^{N} f_{j,s}(t)|\mathbf{j}\rangle$$

where $f_{j,s} = \langle\mathbf{j}|\exp(-iHt)|\mathbf{s}\rangle$

The state of the *r* spin is given by:

$$\rho_{out}(t) = Tr_{1,\ldots,N-1}(|\psi(t)\rangle\langle\psi(t)|) = P(t)(|\psi_{out}\rangle\langle\psi_{out}|) + (1-P(t))|0\rangle\langle 0|$$

$$|\psi_{out}(t)\rangle = \frac{1}{\sqrt{P(t)}}\left(\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)f_{s,r}(t)|1\rangle\right)$$

$$P(t) = \cos^2(\theta/2) + \sin^2(\theta/2)|f_{r,s}|^2$$

The fidelity

A quantity that measures the quality of the state transfer

$$F = \langle \psi_{in} | \rho_{out}(t_o) | \psi_{in} \rangle$$

It makes sense to study the fidelity averaged over all the pure initial states $|\psi_{in}\rangle$ using the Haar measure

$$\bar{F} = \frac{1}{4\pi} \int \langle \psi_{in} | \rho_{out}(t_o) | \psi_{in} \rangle d\Omega$$

In our case:

$$\bar{F} = \frac{|f_{r,s}(t_o)| \cos\gamma}{3} + \frac{|f_{r,s}(t_o)|^2}{6} + \frac{1}{2} \; ; \qquad \gamma = arg\left\{ f_{r,s}(t_o) \right\}$$

The application of Control theory implies that the fields B_i are chosen to maximize the averaged fidelity and that γ is an integer multiple of 2π.

A more specific formulation

The time evolution equation is given by:

$$\dot{X} = -iH(u(t))\ X\ ; \qquad |\psi(t)\rangle = X(t)\ |\psi_o\rangle$$

For systems with dimension n, the operators $X(t)$ are unitary matrices $n \times n$.
In terms of $X(t)$, the system is controllable if for a given time $T < \infty$ and control functions $u(t)$, $X(T)$ can reach any unitary operator.

More precisely:
Let $\bar{U}$ be the functional space that contains $u(t)$
$X(t,u)$ is a solution of $\dot{X} = -iH(u)X$, with control u at time t.
Let $R(T)$ be the set formed by all the unitary matrices $\bar{X}$, with $u \in \bar{U}$ and $X(T,u) = \bar{X}$.
Let $R(\leq T)$ be the set defined as $R(\leq T) := \bigcup_{0 \leq t \leq T} R(t)$.
Then define R as

$$R = \bigcup_{T \geqslant 0} R(T)$$

The systems is called totally controllable if $R = U(n)$, where $U(n)$ is the group of unitary matrices with dimension n.

So, is there a simple answer to the question:

For a given $H(u)$, when $R = U(n)$?

The answer can be obtained introducing the concept off Lie algebra [2]

A Lie algebra L over a filed F is a vector space over F with a "product" $L \times L \longrightarrow L$. For each pair $\{x, y\} \in L$ the bracket $[x, y] \in L$.

[2] *D. D'Alessandro, Introduction to Quantum Control and Dynamics* (Taylor and Francis, Boca Raton, 2008).

Vector space $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with bracket given by

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Matricial Lie algebra, with usual commutator

$$[A, B] := AB - BA$$

In particular:

- $u(n) :=$ Lie algebra of anti-Hermitian or skew-Hermitian matrices $n \times n$.
- $su(n) :=$ Lie algebra of anti-Hermitian matrices with null trace

Controllability criterium

For a given Hamiltonian $H(u(t))$ in a finite Hilbert space and functional space $\bar{U}(n)$, with time evolution equation

$$\dot{X} = -iH(u(t))\ X\ ; \qquad |\psi(t)\rangle = X(t)\ |\psi_o\rangle$$

There is an algorithm that answer the question. It is necessary to generate a Lie algebra $L \in u(n)$ that includes the vector space $span_{u\in\bar{U}}\{-iH(u)\}$ and then check that $L \equiv u(n)$.
To decide that a system is completely controllable just check if $dim(L) = n^2$

What can be done

- Controlled transfer of wave packets defined over several qubits (Gong, Brumer, 2007)
- Perfect state transfer (Di Franco, Paternostro, et al 2008)
- Global controllability with a single local actuator (Schirmer et al, 2008)
- Global control methods for Greenberger-Horne-Zellinger states
- Entanglement between both extremes of a spin chain
- Scalable quantum computation via local control of only two qubits
- ....and so on!